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1987 J. Phys. A: Math. Gen. 20 587

(http://iopscience.iop.org/0305-4470/20/3/020)

J. Phys. A: Math. Gen. 20 (1987) 587-592. Printed in the UK

Rotation interval from a time series

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Received 21 May 1986

Abstract. A very useful concept for two-frequency dynamical systems is the rotation interval. In this paper a method is proposed and tested numerically for estimating it from a time series.

1. Introduction

The rotation interval is a point or closed interval in $\mathbb{R}$, which we write as $[\rho_-, \rho_+]$, allowing $\rho_- = \rho_+$ as a special case, defined (as will be recalled in § 2) for a variety of 'two-frequency' dynamical systems, including:

(i) maps of a circle to itself of degree 1 (not necessarily invertible),

(ii) 'Birkhoff attractors' for dissipative twist maps of a cylinder,

(iii) 'Birkhoff zones of instability' for area-preserving twist maps of a cylinder,

(iv) 3D flows corresponding to (ii) and (iii).

These are relevant to many physical systems, in particular those which can be regarded as two coupled oscillators, or as modulational instabilities on periodic behaviour.

The rotation interval has strong implications for the dynamics. For example, in each of the above cases:

(a) For every rational $p/q \in [\rho_-, \rho_+]$ there is a periodic orbit of rotation number p/q ((i) Newhouse *et al* (1983); (ii) Casdagli (1985), Le Calvez (1985); (iii) Birkhoff (1913).)

(b) For every irrational $\rho \in [\rho_-, \rho_+]$ there is an invariant set of rotation number ρ ((i) Chenciner *et al* (1984); (ii) Casdagli (1985), Le Calvez (1985); (iii) Aubry and Le Daeron (1983), Mather (1982), Katok (1982).)

Furthermore, at least in the case of circle maps:

(c) if the rotation interval is not just a single point then the system has topological chaos (i.e. positive topological entropy) (Block *et al* 1980). This is true for zones of instability too (Boyland 1986, Boyland and Hall 1985), and probably for Birkhoff attractors.

(d) Scaling laws have been predicted (MacKay and Tresser 1986) and verified (Gambaudo *et al* 1985) for the growth of the rotation interval at the transition to topological chaos. The behaviour of the rotation interval depends on the type of route followed, thus providing a footprint for the route.

The only problem with the rotation interval is that its definition depends on the whole system. One orbit does not suffice. Thus it is not clear how to estimate it from a time series, which is all one might get from a physical experiment, for example.

One way to do this, proposed by MacKay and Tresser (1986), is to use the inclusion

$$[\liminf_{n\to\infty} (\theta_n - \theta_0)/n, \limsup_{n\to\infty} (\theta_n - \theta_0)/n] \subset [\rho_-, \rho_+]$$

for all orbits in the appropriate set. This method has the serious drawback, however, that for any invariant measure μ the set of points for which the above limits are different has μ-measure zero, by Birkhoff's ergodic theorem (Birkhoff 1931). Thus if $\rho_- \neq \rho_+$ this method is unlikely to reveal the fact.

Another method, proposed by Casdagli (1986), is to choose an integer $N \approx 10$. Then

$$\rho_- - 1/N \leq \inf_n (\theta_{n+N} - \theta_n)/N \qquad \sup_n (\theta_{n+N} - \theta_n)/N \leq \rho_+ + 1/N.$$

This seems to give results close to optimal, but if one increases N to try and improve the precision, the results deteriorate; the comments of the previous paragraph apply.

Another method proposed by MacKay and Tresser (1986) is to plot the points $(\theta_n - m, \theta_{n+1} - m)$, for $m, n \in \mathbb{Z}$, draw in the least monotone upper bound and greatest monotone lower bound, and calculate their rotation numbers r_+, r_-. Then $[r_-, r_+] \subset [\rho_-, \rho_+]$. But this method suffers from the effect of noise, which broadens $[r_-, r_+]$.

In this paper a simple result is proved, which provides an estimate of the rotation interval from within, and appears to give good estimates without much work.

2. Definitions

Definition 1. For a monotone map $g: \mathbb{R} \to \mathbb{R}$ with $g(\theta+1) = g(\theta)+1$

$$\rho(g) = \lim_{n \to +\infty} (g^n(\theta) - \theta)/n$$

exists for all $\theta \in \mathbb{R}$ and is independent of θ (Poincaré 1885), and is called the *rotation number* of g.

Definition 2. Given a lift f of a degree 1 circle map, $f(\theta+1) = f(\theta)+1$, not necessarily invertible, define its monotone bounds $f_\pm$ by

$$f_+(\theta) = \sup_{\theta' \leq \theta} f(\theta')$$

$$f_-(\theta) = \inf_{\theta' \geq \theta} f(\theta').$$

Then the *rotation interval* of f is defined to be

$$\rho(f) = [\rho(f_-), \rho(f_+)].$$

Equivalently define $g_+(\theta)$ to be the left-most preimage of θ under f and $g_-(\theta)$ to be the right-most preimage. Then $g_\pm$ are monotone and $\rho(f) = [-\rho(g_-), -\rho(g_+)]$.

Definition 3. Given a lift $f: (\theta, r) \to (\theta', r')$ of a dissipative $(0 \leq \det Df \leq \lambda < 1)$ twist map $(\partial\theta'/\partial r \geq \delta > 0)$ of a cylinder with an annular trapping region (i.e. an annulus encircling the cylinder which is mapped into itself), the *Birkhoff attractor* of f is the intersection of all compact, connected invariant sets Σ separating the ends (Birkhoff 1932). Given such a set Σ, take the highest point of each vertical lying in Σ. Then f^{-1} induces a monotone map $g_+ : \theta \mapsto \theta^-$ on these points. Similarly, f^{-1} induces a monotone map g_- on the lowest points of each vertical in Σ. The *rotation interval* of Σ is defined to be $[-\rho(g_-), -\rho(g_+)]$. Note that a circle map f can be regarded as the degenerate case of a dissipative twist map:

$$\theta' = f(\theta) \qquad r' = f(\theta) - \theta$$

and then the definitions of rotation interval agree.

Definition 4. Given an area-preserving ($\det Df = +1$) twist map of a cylinder, a *zone of instability* is an annulus bounded by rotational (i.e. homotopically non-trivial) invariant circles, containing no others. Its *rotation interval* is the interval $[\rho_-, \rho_+]$ between the rotation numbers of the bounding circles.

3. Main result

Theorem. Given an orbit of a circle map, Birkhoff attractor or zone of instability, then

$$\rho_- \leqslant \inf_{m>n} \frac{\lceil \theta_m - \theta_n \rceil}{m-n} \qquad \sup_{m>n} \frac{\lfloor \theta_m - \theta_n \rfloor}{m-n} \leqslant \rho_+$$

where $\lceil x \rceil$ is the least integer $\geqslant x$ and $\lfloor x \rfloor$ is the greatest integer $\leqslant x$.

Proof. The idea in each case is that the integer part of the angle turned through provides bounds on the fastest and slowest rates of advance possible. We give the proofs for the bound on ρ_+. The bound on ρ_- follows similarly.

(i) For a circle map

$$\theta_m - \theta_n \geqslant p \in \mathbb{Z} \Rightarrow f^{m-n}(\theta_n) - p \geqslant 0 \Rightarrow f_+^{m-n}(\theta_n) - p \geqslant 0$$

$$\Rightarrow \rho(f_+^{m-n}) \geqslant p \Rightarrow \rho_+ = \rho(f_+) \geqslant p/(m-n).$$

(ii) For a Birkhoff attractor

$$\theta_m - \theta_n > p \in \mathbb{Z} \Rightarrow \pi_1 f^{-(m-n)}(\theta_m, r_m) + p \leqslant 0$$

where $\pi_1(\theta, r) = \theta$. Since f is a twist map this implies that

$$g_+^{m-n}(\theta_m) + p \leqslant 0.$$

Thus

$$\rho_+ = -\rho(g_+) \geqslant p/(m-n).$$

(iii) For a zone of instability, the argument is the same as for a Birkhoff attractor, using the fact that the boundaries are graphs over θ (Birkhoff 1932). One could use forward iteration just as well as backward in this case.

4. Test

This criterion for the rotation interval was tested for the family of circle maps

$$f_{ab}(\theta) = \theta + a - b/2\pi \sin 2\pi\theta.$$

In figure 1 the results of

(i) calculating the rotation interval by constructing the monotone bounds $f_\pm$ and measuring their rotation numbers using the method described in the appendix,

(ii) estimating the rotating interval, with the initial condition $\theta_0 = 0$, from

$$\Big[\inf_{0\leqslant n<m\leqslant 100} \lceil \theta_m - \theta_n \rceil/(m-n), \quad \sup_{0\leqslant n<m\leqslant 100} \lfloor \theta_m - \theta_n \rfloor/(m-n)\Big]$$

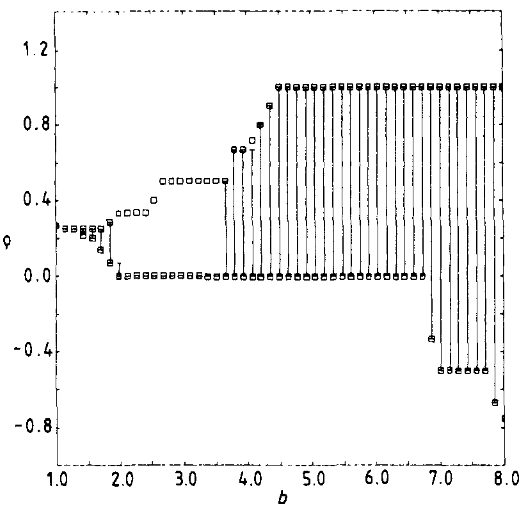

Figure 1. Rotation interval for the family of circle maps

$$f_{ab}(\theta) = \theta + a - b/2\pi \sin 2\pi\theta$$

with $a = 0.3$, $b = 1.0$ to 8.0, measured by finding the rotation numbers of the monotone bounds (□), and estimated using the criterion of § 3 with initial condition $\theta_0 = 0$ (vertical bars).

are compared, for $a = 0.3$, $b = 1.0$ to 8.0 in 50 steps. It appears that for many parameter values, the two are equal. Sometimes the estimated interval was smaller than the true interval, because the chosen orbit did not explore the available space sufficiently. Usually, the trouble was that it got attracted to a periodic orbit. A different initial condition often reveals more of the rotation interval, e.g. figure 2 is for the same parameter values as figure 1 but with $\theta_0 = 0.5$.

It would be interesting to test this method also on Birkhoff attractors (cf Casdagli 1986), and on zones of instability.

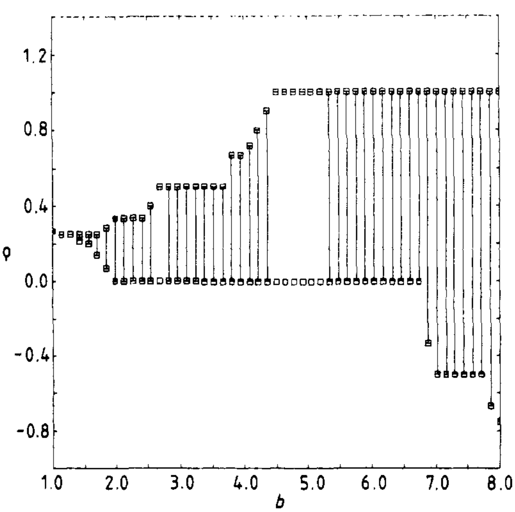

Figure 2. The same as figure 1 for initial condition $\theta_0 = 0.5$.

5. Practical considerations

To use this method on time series from physical experiments, one must first be sure that one has a 'two-frequency' system. If the system has two obvious oscillations, e.g. periodically forced pendulum, then it is clear.

Otherwise, a general procedure to test this is to form a surface of section plot. Measure two variables (x, y) every time a third variable z passes through a chosen phase, e.g. positive zero crossing. Plot the points (x_n, y_n) in the plane. If they form something roughly circular then you are in business. If not, you may just need to change your choice of variables x and y to get a better projection. They should be roughly 90° out of phase. You might get away with measuring one variable x and using the previous value of x for y, or some weighted average of the previous few values of x.

The next step is to extract an angle variable θ_n from (x_n, y_n). If the surface of section plot is roughly circular this should be fairly straightforward, e.g. choose an origin O inside the circle and measure the angle θ_n of the ray from O to (x_n, y_n). You must be careful to get the correct integer part of θ_n. It is easiest to do if the variable z is the fastest oscillating variable. Otherwise you may have to measure x and y at some intermediate phases of z in order to count how many revolutions have been made.

Finally, some cautionary remarks.

(i) To apply the criterion for a two-dimensional map one needs twist. This is probably almost always satisfied in practice for some choice of angle coordinate. The only problem is that you might not have a good choice.

(ii) The Birkhoff attractor can coexist with other attractors. Thus the rotation interval one measures might have nothing to do with the Birkhoff attractor.

(iii) One must let transients die away to be sure that one is in the Birkhoff attractor, before estimating its rotation interval. Actually Birkhoff attractors are not necessarily attracting, so this could be a problem.

(iv) Birkhoff attractors do not necessarily have a dense orbit, so they can contain a lot of transient behaviour and the estimated rotation interval may be much smaller than the true one.

I intend to test this method of measuring the rotation interval on a system of two coupled electronic oscillators.

Acknowledgments

This paper was written while visiting the University of Arizona, Tucson, for their special year on Chaos and Turbulence, supported by the US Office of Naval Research under grant N0014-85-K-0412. I would like to thank Martin Casdagli for stimulating conversations and comments and for sharing his results before publication, and the UK Science and Engineering Research Council and the Nuffield Foundation for computing facilities.

Appendix

For a monotone map $g:\mathbb{R}\to\mathbb{R}$ with $g(\theta+1)=g(\theta)+1$, the following algorithm estimates the rotation number very efficiently. At every stage $\rho(g)\in[p_-/q_-, p_+/q_+]$, and this interval has width $1/q_-q_+$.

Choose any θ_0, evaluate $\theta_1 = g(\theta_0)$, and let $q_\pm = 1$, $p_- = \lfloor \theta_1 \rfloor$, $p_+ = \lceil \theta_1 \rceil$. Repeat the following until $q_- q_+$ is large enough.

Evaluate $\theta_{q_+ + q_-} - p_+ - p_- - \theta_0$.

If it is zero then $\rho(g) = (p_+ + p_-)/(q_+ + q_-)$; exit.

If it is positive replace q_+, p_+ by $q_+ + q_-$, $p_+ + p_-$.

If it is negative replace q_-, p_- by $q_+ + q_-$, $p_+ + p_-$.

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